A Tour of Social Choice Theory: Exercise Solutions

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Last year, I wrote the article [1] for the excellent recreational mathematics journal Parabola. Included within that article were a series of exercises to guide the reader through a proof of May's theorem. This note includes solutions to these exercises, and assumes the reader is familiar with all the relevant definitions from [1, Section 2]. As with any set of problems, the fun is in solving them for yourself, but for anyone who has done them and wants to compare their solutions against mine, I have included mine below.

Exercise 1. *Show that the majority vote is anonymous, neutral and positively responsive.*[1]

Proof. Let f be a majority vote function.

Since μ counts the number of 1 indices, it is invariant under permutations. Therefore if $\vec{x}$ is a permutation of $\vec{y}$, $\mu(\vec{x}) = \mu(\vec{y})$. Since f is function of the value of μ, $f(\vec{x}) = f(\vec{y})$, and f is anonymous.

Suppose $f(\vec{x}) = 1$. This means that $-f(\vec{x}) = -1$ and $\mu(\vec{x}) > n/2$, which happens if and only if $\mu(-\vec{x}) - n = -\mu(\vec{x}) < -n/2$, or equivalently if $\mu(-\vec{x}) < n/2$. That means that $f(-\vec{x}) = -1$. By symmetry, replacing $\vec{x}$ by $-\vec{x}$ we obtain the converse. When $f(\vec{x}) = 0$ the result follows in the same way. Therefore f is neutral.

Suppose $\vec{x} = (x_1, \dots, x_n)$ and $\vec{y} = (y_1, \dots, y_n)$ satisfying $x_k \geq y_k$ for all $k \in \{1, \dots, n\}$. Thus if $y_k = 1$, $x_k = 1$, and $\mu(\vec{x}) \geq \mu(\vec{y})$, meaning f is positively responsive. □

Exercise 2. *Prove that for all $\vec{x} \in \{-1, 1\}^n$,*[2]

(a) $\mu(\vec{x}) + \mu(-\vec{x}) = n$,

(b) $\mu(\vec{x}) > \frac{n}{2} \iff \mu(-\vec{x}) < \frac{n}{2}$,

(c) $\mu(\vec{x}) > \mu(-\vec{x}) \iff \mu(\vec{x}) > \frac{n}{2}$ *and* $\mu(\vec{x}) < \mu(-\vec{x}) \iff \mu(\vec{x}) < \frac{n}{2}$,

(d) $\mu(\vec{x}) = \frac{1}{2}\sum_{i=1}^{n} x_i + \frac{n}{2}$.

Proof.

(a) Since $\mu(\vec{x})$ counts the number of components of $\vec{x}$ which are 1, $\mu(-\vec{x})$ counts the number of components of $-\vec{x}$ which are 1, or equivalently components of $\vec{x}$ which are -1. Since each component is either 1 or -1, these two quantities sum to the length of this vector, so

$$\mu(\vec{x}) + \mu(-\vec{x}) = n.$$

[1] In retrospect, I should have placed this question after the next exercise on μ, since having derived simple properties of μ makes it easier.

[2] In [1] this exercise has a typo, where instead of $\{-1, 1\}^n$ it was written as $\{-1, 1\}$.

(b) Using the previous part, we can see that

$$\begin{aligned}\mu(\vec{x}) > \frac{n}{2} &\iff n - \mu(-\vec{x}) > \frac{n}{2}\\ &\iff -\mu(-\vec{x}) > -\frac{n}{2}\\ &\iff \mu(-\vec{x}) < \frac{n}{2}.\end{aligned}$$

(c) The first implication is a consequence of the first part as follows

$$\begin{aligned}\mu(\vec{x}) > \mu(-\vec{x}) &\iff \mu(\vec{x}) > n - \mu(\vec{x})\\ &\iff 2\mu(\vec{x}) > n\\ &\iff \mu(\vec{x}) > \frac{n}{2}.\end{aligned}$$

Replacing $\vec{x}$ with $-\vec{x}$ and using the second part, we see that

$$\mu(\vec{x}) < \mu(-\vec{x}) \iff \mu(-\vec{x}) > \frac{n}{2} \iff \mu(\vec{x}) < \frac{n}{2}.$$

(d) Starting with the right hand side, we notice that

$$\frac{1}{2}\sum_{i=1}^{n} x_i + \frac{n}{2} = \sum_{i=1}^{n} \frac{1}{2}(x_i + 1).$$

The inner term $(x_i + 1)/2$ is 1 when $x_i = 1$ and 0 otherwise, and hence counts the number of integers $i \in \{1, \ldots, n\}$ such that $x_i = 1$. This is $\mu(\vec{x})$ by definition.

□

Exercise 3. *Suppose $f : \{-1, 1\}^n \to \{-1, 0, 1\}$ is an anonymous and positively responsive social choice function . If for some $\vec{x}, \vec{y} \in \{-1, 1\}^n$, $\mu(\vec{x}) \geq \mu(\vec{y})$, then $f(\vec{x}) \geq f(\vec{y})$.*

Proof. Suppose $\vec{x}, \vec{y} \in \{-1, 1\}^n$ such that $\mu(\vec{x}) \geq \mu(\vec{y})$. Permute $\vec{x}$ into the vector $\vec{x}'$ by shuffling all components which are 1 to the left, and likewise with $\vec{y}$ into $\vec{y}'$.

Notice that μ is invariant under permutation as we do not change the number of components with each value, just their positons, and hence

$$\mu(\vec{x}') \geq \mu(\vec{y}').$$

Moreover, because these vectors are sorted, either $x'_i = y'_i$ or $x'_i > y'_i$ for each $i \in \{1, ..., n\}$. By monotonicity of f,

$$f(\vec{x}') \geq f(\vec{y}'),$$

and then applying anonymity, we may conclude that

$$f(\vec{x}) = f(\vec{x}') \geq f(\vec{y}') = f(\vec{y}),$$

as required. □

Exercise 4. *Suppose $f : \{-1,1\}^n \to \{-1,0,1\}$ is an anonymous, neutral and positively responsive social choice function. For all $\vec{x} \in \{-1,1\}^n$,*

(a) if $\mu(\vec{x}) > \frac{n}{2}$, then $f(\vec{x}) \in \{0,1\}$,

(b) if $\mu(\vec{x}) < \frac{n}{2}$, then $f(\vec{x}) \in \{-1,0\}$,

(c) if $\mu(\vec{x}) = \frac{n}{2}$, then $f(\vec{x}) = 0$.

Proof. Suppose $f : \{-1,1\}^n \to \{-1,0,1\}$ is an anonymous, neutral and positively responsive social choice function.

Take an $\vec{x} \in \{-1,1\}^n$ such that $\mu(\vec{x}) > \frac{n}{2}$. By Exercise 2 (c),

$$\mu(\vec{x}) > \mu(-\vec{x}),$$

and so by Exercise 3

$$f(\vec{x}) \geq f(-\vec{x}).$$

By neutrality, we may deduce that

$$f(\vec{x}) \geq f(-\vec{x}) = -f(\vec{x}) \implies 2f(\vec{x}) \geq 0$$

and so

$$f(\vec{x}) \in \{0,1\}.$$

For the second part, we may apply the previous part with $-\vec{x}$ which lets us deduce that

$$\mu(-\vec{x}) > \frac{n}{2} \implies f(-\vec{x}) \in \{0,1\}.$$

By Exercise 2 (b) the first condition is equivalent to

$$\mu(\vec{x}) < \frac{n}{2},$$

while by neutrality, the second condition is equivalent to

$$f(\vec{x}) \in \{0,-1\}.$$

Finally, if

$$\mu(\vec{x}) = \frac{n}{2},$$

then by Exercise 2 (a)

$$\mu(-\vec{x}) = \frac{n}{2}.$$

Using Exercise 3, since $\mu(\vec{x}) \leq \mu(-\vec{x})$ and $\mu(-\vec{x}) \leq \mu(\vec{x})$, we may conclude that $f(\vec{x}) = f(-\vec{x})$. Using neutrality

$$f(\vec{x}) = f(-\vec{x}) = -f(\vec{x})$$

and so $f(\vec{x}) = 0$. $\square$

Exercise 5. *Prove May's Theorem. That is, suppose $f : \{-1, 1\}^n \to \{-1, 0, 1\}$ is an anonymous, neutral and positively responsive social choice function, and prove that:*

(a) if f is chosen to minimise the number of possible ties, then f is the majority vote,

(b) if n is odd and f cannot result in a tie, then f must be the majority vote,

(c) if n is even and f cannot result in a tie, then no such f exists.

Proof.

(a) First suppose that f is chosen to minimise the number of ties. We know, by Exercise 4 () that if $\mu(\vec{x}) = n/2$, then $f(\vec{x}) = 0$. This means that a tie must at least occur when $\mu(\vec{x}) = n/2$. The majority vote yields ties precicely under this condition, and thus because, by Exercise 1, the majority vote satisfies the assumptions of the theorem, it is the unique such function which minimises ties.

(b) Suppose that n is odd and $f(\vec{x}) \neq 0$ for all profiles $\vec{x}$. Now, notice that if $f(\vec{x}) \notin \{-1, 0\}$ then $\mu(\vec{x}) \geq n/2$ by Exercise 4 (b). However because $\mu(\vec{x})$ is an integer, and n is odd, this second condition is equivalent to $\mu(\vec{x}) > n/2$. Thus $f(\vec{x}) = 1$ implies $\mu(\vec{x}) > n/2$. A similar argument shows that $f(\vec{x}) = -1$ implies $\mu(\vec{x}) < n/2$. Because $f(\vec{x}) \neq 0$ and n is odd, taking the contrapositive of each of these statements uniquely defines f as the majority vote.

(c) If n is even, and $f(\vec{x}) \neq 0$ for all profiles $\vec{x}$. Choosing a profile $\vec{x}$ with exactly $n/2$ votes for 1 and $n/2$ votes for -1, then $\mu(\vec{x}) = n/2$, and we force a tie by Exercise 4 (a).

□

References

[1] A. Manning, 'A Tour of Social Choice Theory', *Parabola* **61(2)**, (2025).